Assignment 7

This homework is due Friday March 18.

There are total 42 points in this assignment. 38 points is considered 100%. If you go over 38 points, you will get over 100% for this homework (but not over 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should exhibit your work and contain full proofs. Bare answers will not earn you much.

This assignment covers Sections 5.1–5.3 of Textbook.

(1) [5pt] Express e^z in the form $u + iv$ for the following z.

(a) $-\frac{\pi}{3}$.
(b) $\frac{1}{2} - i\frac{\pi}{4}$.
(c) $-4 + 5i$.
(d) $\frac{\pi}{3} - 2i$.
(e) $-1 + i\frac{3\pi}{2}$.

(2) [2pt] Use the fact that e^{z^2} is analytic to show that $e^{x^2-y^2}\sin 2xy$ is harmonic.

(3) [10pt] Show the following concerning the exponential map.
(a) The image of the first quadrant $\{(x, y) : x > 0, y > 0\}$ is the region $\{w : |w| > 1\}$.
(b) If a is a real constant, the horizontal strip $\{(x, y) : a < y \le a + 2\pi\}$ is mapped one-to-one and onto all nonzero complex numbers.
(c) The image of the vertical line segment $\{(x, y) : x = 2, y = t\}$, where $\frac{\pi}{6} < t < \frac{7\pi}{6}$, is half a circle.
(d) The image of the horizontal ray $\{(x, y) : x > 0, y = \frac{\pi}{3}\}$ is a ray.

(4) [5pt] Find all values of the following. (Reminder: $\log z$ is a multivalued function, $\operatorname{Log} z$ is its principal branch.)

(a) $\operatorname{Log}(ie^2)$,
(b) $\operatorname{Log}(\sqrt{3} - i)$,
(c) $\operatorname{Log}((1 + i)^4)$,
(d) $\log(-3)$,
(e) $\log(-\sqrt{2} + i\sqrt{2})$,

(5) [2pt] Give an example of specific values of z_1, z_2 such that
$\operatorname{Log}\left(\frac{z_1}{z_2}\right) \neq \operatorname{Log}(z_1) - \operatorname{Log}(z_2)$.

(6) [5pt] Solve the following equations (i.e. find all possible values of z).

(a) $\operatorname{Log}(z) = 1 - i\frac{\pi}{4}$.
(b) $\operatorname{Log}(z - 1) = i\frac{\pi}{2}$.
(c) $\exp(iz) = -1$.
(d) $\exp(z + 1) = i$.

(7) [3pt] Find the principal value of
(a) 4^i.
(b) $(-1)^{\frac{1}{\pi}}$.
(c) $(1 + i\sqrt{3})^{\frac{i}{2}}$.

(8) [5pt] Find all values of the expressions below. In each case determine if there are infinitely many or finitely many values.

(a) $(-i)^i$.
(b) $(-1)^{\sqrt{2}}$.
(c) $(-1)^{\frac{3}{4}}$.
(d) $(1 + i)^{2-i}$.

(9) [5pt] For $z = re^{i\theta} \neq 0$, show that for $r > 0$ and $-\pi < \theta \le \pi$, the principal branch of the function
(a) z^i is given by $z^i = e^{-\theta}(\cos(\ln r) + i\sin(\ln r))$.
(b) z^α (with real α) is given by $z^\alpha = r^\alpha(\cos\alpha\theta + i\sin\alpha\theta)$.
(*Hint:* Use the definition of the power function.)